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Application of 2D-infinite beam elements in dynamic analysis of train-track interaction †

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Abstract

Railway structure is a structure with an infinite length for which its modeling is usually carried out by accepting some assumptions for boundary conditions. In practice, there are various methods of modeling by forming equations of motions and dynamic analysis. One of these methods is modeling the railway track by finite element method. Generally in such modeling, a limited length of a track is modeled and executed by implementing some boundary conditions. Assortment of boundary conditions has an effect on dynamic responses. To reduce such effects, usually the length of the track is chosen to such a measure to minimize the dynamic responses of the end points of the track elements to zero. Doing so would cause an increase in the length of the track and hence add to an equation's degrees of freedom, volume of output and prolongation of analysis time. For this reason, a combination of finite elements and infinite beam elements (two end elements) has been proposed for railway track modeling. Also, matrices of mass, damping and stiffness of one infinite element which has been laid on a visco-elastic bed, have been calculated by implementing selected shape functions. Therefore, by applying two infinite beam elements on either side of the model, a railway track is formed like a beam on an elastic bed which creates the possibility of eliminating the boundary condition effects.

Keywords: Boundary condition; Dynamic train - track interaction; Infinite beam element; Railway track

1. Introduction

Generally, in analyzing the dynamic behavior of three dimensional structures, particularly when considering the soil-structure interaction or structure–soil-water interaction, modeling of continuous soil environment is necessary in order to illustrate the real behavior [1, 2]. Dynamic analysis of structures occurs in frequency domain or time domain. Where, analysis occur in time domain by applying numerical methods, such as finite elements, it is necessary to mesh a limited length of the structure, form their equation of motion and after executing boundary conditions reach the results in the form of time history. In this kind of analysis, the effects of boundary conditions are effective in the attained results. Bettess [3] and Zinkiewicz [4] have applied finite element method in such analysis. Practically, in this analysis a combination of finite elements together with a few infinite or semi infinite elements have been utilized at the boundary points. Infinite elements shall be assumed in such a way that the material used shall continue indefinitely and their deformation shall become zero at infinity (real situation).

Another engineering problem that has an unlimited length and applying boundary conditions in the time domain has an effect on its results, is the problem related to dynamic analysis of railway tracks. Duffy [5], Patil [6] and Cai [7, 8] have investigated the effects of moving masses on an infinite beam (rail). Also Zakeri & Xia [9] have examined the railway

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track by applying 2D-infinite beam element model. Several 2D infinite element models of a beam have been presented [10]. In this paper details related to mapping of an infinite element to a finite element (Fig. 1) have been illustrated and mass matrix, damping and stiffness have been calculated.

2. Mapping function

Fig. 1 demonstrates the general principal of the mapping function. To begin with, a semi-infinite beam CPQ is assumed for mapping in the k_{eq}, c_{eq} direction of the "x" axis. So it can be written:

$$x = -\frac{\xi}{1-\xi}x_c + (1+\frac{\xi}{1-\xi})x_Q \tag{1}$$

Where:

$\xi = 0$ corresponds to $x = x_Q$

$\xi = 1$ corresponds to $x = \infty$

$\xi = -1$ corresponds to $x = \frac{x_Q + x_c}{2} = x_p$

Where x_p is an imaginary point in the middle of the infinite or finite beam.

Assuming $x_p = a$ and $x_Q = 2a$ ("a" assumed as 10 sleepers spacing.)

$$x = \frac{2a}{1-\xi}, \quad \xi = 1 - \frac{2a}{x} \tag{2}$$

According to the finite element method for a special element PR:

$$[u]^e = Na \tag{3}$$

In this special case (railway track), from "strength of material" theory, the strains are expressed as the second derivatives of displacement. Of course, one must be sure of continuity in displacement and its gradient between the elements. Considering w and

$$w_x \equiv \frac{dw}{dx} = \theta$$

as displacement and the gradient of point x, one can write:

$$[u]_i = \begin{Bmatrix} w_i \\ w_{xi} \end{Bmatrix} \tag{4}$$

Assuming three nodes (with six variables), one can define the below shape functions with two coefficients.

$$w = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + a_6 x^5 \tag{5}$$

By changing the variable x and ξ, one can describe the shape functions as below (Fig. 2):

$$\begin{aligned} N_1(\xi) &= \frac{1}{4}(4+3\xi)\xi^2(\xi-1)^2 \\ N_2(\xi) &= \frac{1}{4}(1+\xi)\xi^2(\xi-1)^2 \\ N_3(\xi) &= (1-4\xi)(\xi+1)^2(\xi-1)^2 \\ N_4(\xi) &= \xi(\xi+1)^2(\xi-1)^2 \\ N_5(\xi) &= \frac{1}{4}(4-3\xi)\xi^2(\xi+1)^2 \\ N_6(\xi) &= \frac{1}{4}(\xi-1)\xi^2(\xi+1)^2 \end{aligned} \tag{6}$$

The above six shape functions have been sketched in Fig. 3 which can be used in determining the mass matrix and stiffness.

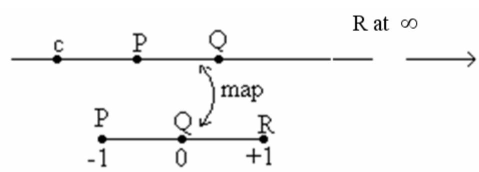

Fig. 1. Mapping of infinite element to finite element.

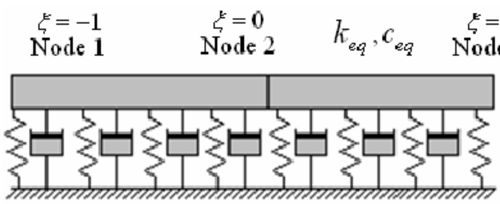

Fig. 2. Infinite rail beam on visco-elastic foundation mapped to beam element.

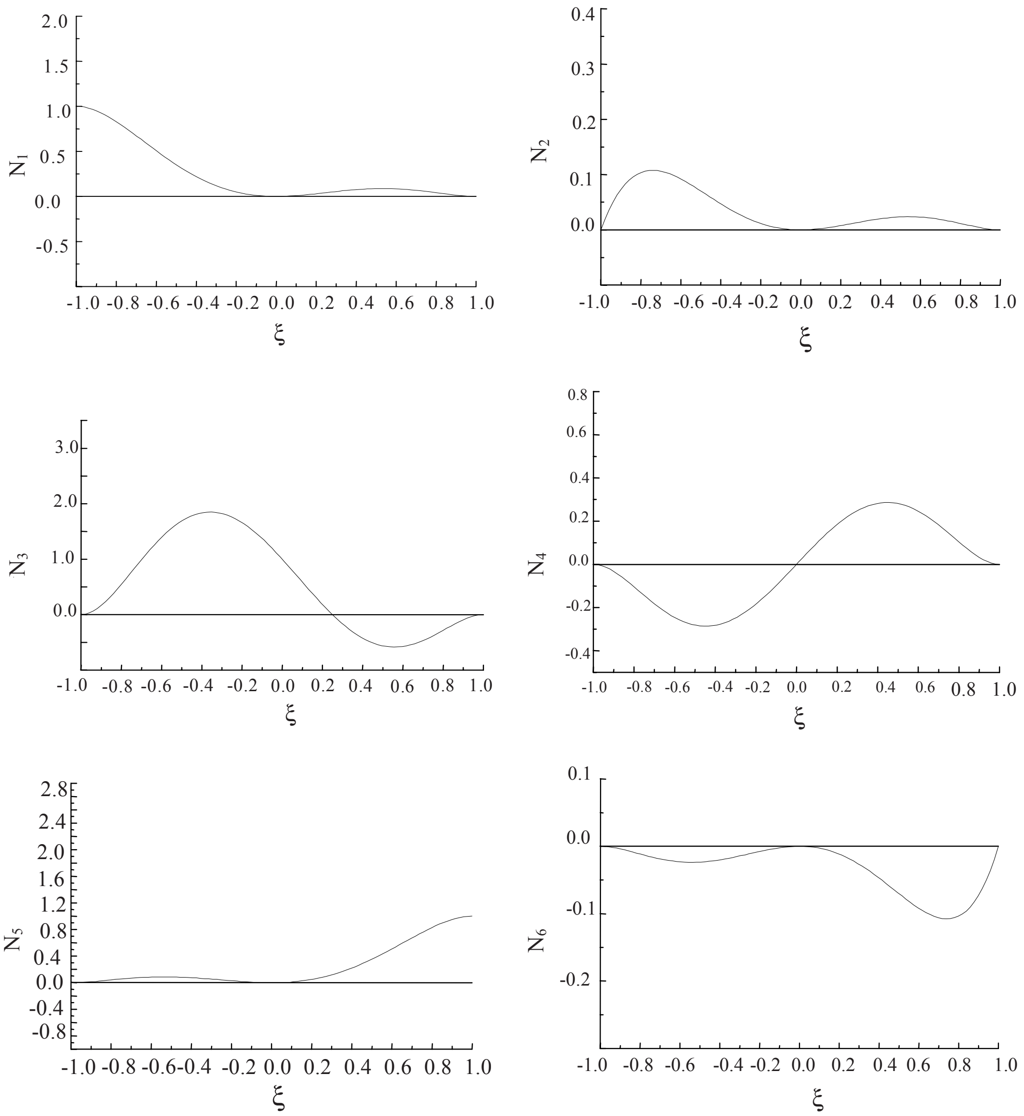

Fig. 3. Shape functions of infinite beam element.

3. Calculation of mass matrix

The second derivatives of the above shape functions are implemented in the calculation of stiffness matrix coefficient. For a beam with three nodes, the vertical displacement of any section with a distance of the “x” along the length of the beam can be expressed as follows:

$$w(x) = \sum_{j=1}^{6} N_j u_j = [N]\{u\}^e$$
$$\{u\}^e = [w_1, w_{x1}, w_2, w_{x2}, w_3, w_{x3}]^T \tag{7}$$

Considering the displacement and the gradients of the rail at infinity (third node) is zero, the above equation can be written in the following form:

$$w(x) = \sum_{j=1}^{4} N_j u_j = [N]_4 \{u\}_4^e \tag{8}$$
$$[N]_4 = [N_1, N_2, N_3, N_4]$$
$$\{u\}^e = [w_1, w_{x1}, w_2, w_{x2}]^T$$

Before the calculations of mass matrixes and the required stiffness, it is necessary to mention that; the velocity of any section has been characterized by the parameter $\dot{w}$ and the kinetic energy of the beam is described as follows:

$$\begin{aligned} T &= \frac{m}{2}\int_{X_1}^{\infty}\frac{1}{2}m_r\dot{w}^2(x)dx \\ &= \frac{1}{2}\{\dot{u}\}^{eT}[\int_{X_1}^{\infty} m_r[N(x)]^T[N(x)]dx]\{\dot{u}\}^e \\ &= \frac{1}{2}\{\dot{u}\}^{eT}[M]^e\{\dot{u}\}^e \end{aligned} \tag{9}$$

Where:

$$\begin{aligned} &[M]^e = m_r\int_{X_1}^{\infty}[N(x)]_4^T[N(x)]_4 dx \\ &\text{or} \quad [M]^e = m_r[R_1] \\ &[R_1] = \int_{X_1}^{\infty}[N(x)]_4^T[N(x)]_4 dx \\ &\qquad = \int_{-1}^{1}[N(\xi)]_4^T[N(\xi)]_4 |J_b| d\xi \\ &|J_b| = \frac{2a}{(1-\xi)^2} \end{aligned} \tag{10}$$

By integration of these equations, one can write:

$$[R_1] = \frac{a}{315}\begin{bmatrix} 72 & 10.5 & 0 & 24 \\ 10.5 & 2 & -8 & 8 \\ 0 & -8 & 1280 & -320 \\ 24 & 8 & -320 & 128 \end{bmatrix} \tag{11}$$

So, the mass matrix can be calculated easily as follows:

$$\begin{aligned} [M]^e &= m_r[R_1] \\ &= \frac{m_r a}{315}\begin{bmatrix} 72 & 10.5 & 0 & 24 \\ 10.5 & 2 & -8 & 8 \\ 0 & -8 & 1280 & -320 \\ 24 & 8 & -320 & 128 \end{bmatrix} \end{aligned} \tag{12}$$

4. Calculations of stiffness and damping matrices

As can be seen on Fig. 2, some elements of an infinite beam are placed on an elastic bed. So, the damping and stiffness matrices would have two independent parts. The first part arises from stiffness and damping of the element's material and the second part from the elastic bed. Therefore:

$$[K]^e = [K_s]^e + [K_f]^e \tag{13}$$

The first part, the stiffness matrix of a beam element would be determined by considering its potential energy and with reference to mechanics of structures, one can write:

$$\begin{aligned} U_s &= \int_{x_1}^{\infty}\frac{1}{2}EI[w''(x)^2]dx \\ &= \frac{1}{2}\{u\}^{eT}\int_{x_1}^{\infty}EI[N'']^T[N'']dx\{u\}^e \\ U_s &= \frac{1}{2}\{u\}^{eT}[K_s]^e\{u\}^e \\ [K_s]^e &= \int_{x_1}^{\infty}EI[N'']^T[N'']dx \\ [K_s]^e &= EI[R_2] \\ [R_2] &= \int_{x_1}^{\infty}[N''(x)]^T[N''(x)]dx \\ [R_2] &= \int_{-1}^{1}|J_b|[N''(x)]^T[N''(x)]d\xi \\ N''_j &= \frac{d^2N}{dx^2} = \frac{1}{J^3}(\frac{d^2N}{d\xi^2}J_b - \frac{dN}{d\xi}\frac{dJ_b}{d\xi}) \end{aligned} \tag{14}$$

Substituting shape functions of Eq. (6) in these equations and integrating them, one can write:

$$[K_s] = \frac{EI}{3003a^3}\begin{bmatrix} 221536 & 135786 & -725312 & 139104 \\ 135786 & 102488 & -407200 & 74656 \\ -725312 & -407200 & 2578320 & -488960 \\ 139104 & 74656 & -488960 & 97280 \end{bmatrix} \tag{14}$$

The second part, stiffness matrix of a beam element would be determined by considering stiffness of the bed K_{eq} and using the potential energy of the beam (rail). Again, with reference to mechanics of structures:

$$\begin{aligned} U_f &= \int_{x_1}^{\infty}\frac{1}{2}k_{eq}[w(x)^2]dx \\ &= \frac{1}{2}\{u\}^{eT}\int_{x_1}^{\infty}k_{eq}[N]^T[N]dx\{u\}^e \\ U_f &= \frac{1}{2}\{u\}^{eT}[K_f]^e\{u\}^e \\ [K_f]^e &= \int_{x_1}^{\infty}k_{eq}[N]^T[N]dx \\ [K_f]^e &= k_{eq}[R_1] \end{aligned} \tag{15}$$

So,

$$[K_f]^e = k_{eq}[R_1] = \frac{k_{eq}a}{315}\begin{bmatrix} 72 & 10.5 & 0 & 24 \\ 10.5 & 2 & -8 & 8 \\ 0 & -8 & 1280 & -320 \\ 24 & 8 & -320 & 128 \end{bmatrix} \quad (16)$$

Finally, combining the two parts, determines the stiffness matrix.

$$[K]^e = [K_s]^e + [K_f]^e \quad (17)$$

Similarly and by using mass and stiffness matrixes, the damping matrix is as follows:

$$[C]^e = [C_s]^e + [C_f]^e = \alpha[M]^e + \beta[K_s]^e + c_{eq}[R_1] \quad (18)$$

5. Effects of track length and boundary conditions on the responses of railway tracks: case study

Two parameters, length of selected model and boundary conditions; play an important role in the railway track's dynamic response in time domain. Selecting a long portion of track in modeling of a railway track and analyzing the dynamic of the whole system would prolong the process period due to enormous size of the matrices and would make it all uneconomical. Creating unrealistic boundary constraints with regard to boundary conditions would have an effect on the dynamic response of the track elements, particularly on the "rail vibration acceleration" and "reflection of rail accelerations," hence giving inaccurate results.

Revising the technical contexts of [7, 11, 12] illustrates that; a moving mass of 20-30 sleepers (11 m to 15 m) has been used in the modeling of a track with simple boundary conditions or constrained or elastic for once.

In this paper, the dynamic response of the track at a specified point (middle point) has been calculated by considering a moving rail vehicle on the track and the results having varied the length of the model have been presented, all by implementing the infinite boundary elements and real boundary conditions. This model has been illustrated in Fig. 4. The dynamic analysis was carried out by applying the computer program DATI, which is described in reference [12].

The calculated results are well in accordance, both in response curves, in amplitudes and in distribution tendencies, with the existing experimental data [9], which verifies the effectiveness of the analytical model and the computer simulation method. Since the interaction problem is generally solved by use of numerical time-integration, the computer time when employing this type of track model substantially exceeds the time required in an analysis of less detailed models.

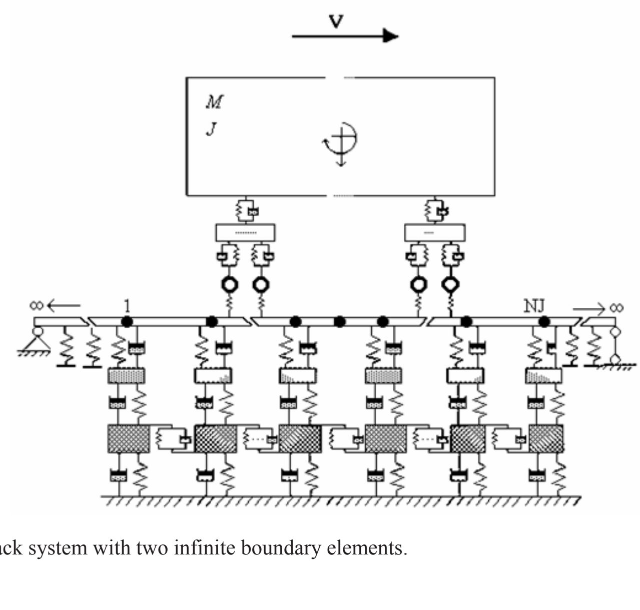

Fig. 4. Model of train – track system with two infinite boundary elements.

In the first step, the minimum length of the model has been determined, considering the rigid wheel base length of 21.5 m of the vehicle and 41 sleepers spacing (22.5 m). A speed of 160 Km/h has been considered and the responses have been measured at the middle of the model. The results show that; the effects of unrealistic boundary conditions in rail vibration acceleration would become clear in time between passing two bogies. Clearly selecting a shorter length model (same length as the vehicle), and unrealistic boundary conditions would cause to reflect and return the waves and thus increase the vibration acceleration. So, selecting a model shorter than the vehicle length is insufficient. Calculations have been continued for the lengths of 25, 27.5, 29, 32, 35, 40, 45, 50, 55 and 60. The maximum differences between the rail vibration accelerations have been compared and the results reached are in Fig. 5.

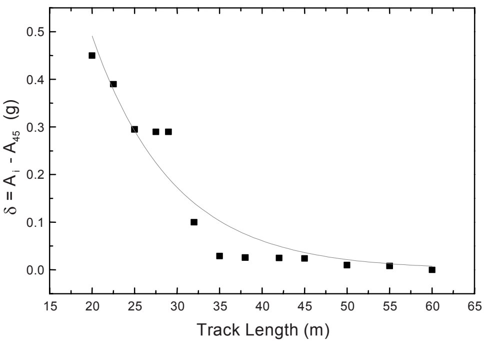

Fig. 5. Variation of rail acceleration in track models with different length.

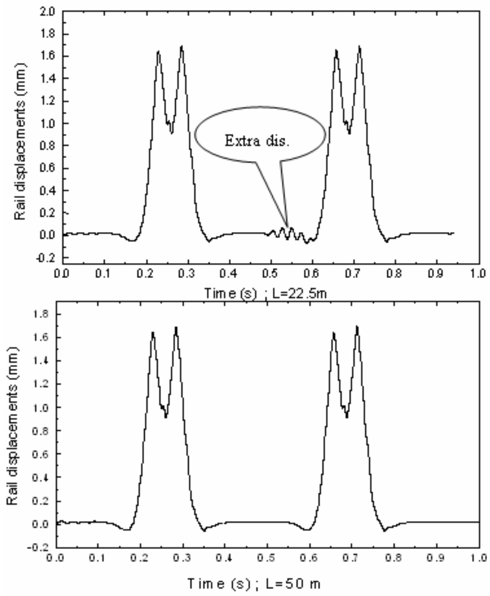

Fig. 6. Displacement time history of mid point in two different models.

Related to the time history of rail displacement, it can be said that; selecting a shorter length and unrealistic boundary conditions, would make their effects apparent before the second crossing. The results of two different lengths and boundary conditions are shown in Fig. 6.

6. Conclusion

A case study together with dynamic analysis of track-train interaction has been conducted. The track structure model is of Timoshenko beam on discrete elastic supports. The discrete elastic supports have been distinguished by modeling sleeper and ballast as a concentrated (lumped) masses joined by springs and dampers. A body for the vehicle together with two bogies each having two degrees of freedom has been assumed with two configurations of four or six axels. Considering a rigid wheel base length of the vehicle of about 21.5 m, the calculations were performed for two separate cases:

A) With constraint boundary conditions having the length of 22.5 m to 60 m.

B) By using two infinite beam elements and a general length of bigger than 1.5 times the vehicle length.

Both ways' results have been compared. Meantime the description and the calculation of infinite element for the two ends of the track was carried out, and the track-train's dynamic system was modeled and analyzed. As can be seen from the results, changes in the maximum value of displacement are not considerable, but in the case when the constrained boundary conditions have been executed, disorder in the displacement of the mid span, which is the result of returned waves and the reflection of rail accelerations, occurs. The results show that when the model length is shorter than 1.5 times the vehicle length, the effects of unrealistic boundary conditions in rail vibration acceleration would become clear in time between passing two bogies. Also, by choosing the model length 1.5 times the vehicle length, the effects of boundary condition is at its minimum and disorders in time history of rail acceleration and displacement, especially before the second bogie crosses disappears.

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